

Critical RSOS Models in External Fields

I. Vaysburd

International School for Advanced Studies (SISSA/ISAS)

Via Beirut 2, 34014 Trieste, Italy

vaysburd@tsmi19.sissa.it

Abstract

We suggest a new family of unitary RSOS scattering models which is obtained by placing the $SO(N)$ critical models in "electric" or "magnetic" field. These fields are associated with two operators from the space of the $SO(N)$ RCFT corresponding to the highest weight of the vector representation of $SO(N)$. A perturbation by the external fields destroys the Weyl group symmetry of an original statistical model. We show that the resulting kinks scattering theories can be viewed as affine imaginary Toda models for non-simply-laced and twisted algebras taken at rational values (roots of unity) of q -parameter. We construct the fundamental kink S -matrices for these models. At the levels $k = 1, 2, \infty$ our answers match the known results for the Sine-Gordon, Z_{2N} - parafermions and free fermions respectively. As a by-product in the $SO(4)$ -case we obtain an RSOS S -matrix describing an integrable coupling of two minimal CFT.

1 Introduction

The restricted-solid-on-solid models (RSOS) [1] - [3] form an important and interesting class of integrable models. They appear in various mathematical and physical topics such as statistical mechanics, conformal field theory (CFT), quantum groups (QG) et.c. A large class of the $2D$ statistical models (including Ising, Z_N - models and their multicritical versions) can be reformulated in RSOS terms. It means that one can attach a finite Lie algebra \mathcal{G} to each of these models. Dynamical variables λ placed at sites of a $2D$ - lattice belong to a subset of the integral positive weights of the algebra \mathcal{G} marking the irreducible representations π_λ . The subset is constrained by

$$\lambda\theta \leq k \quad (1)$$

where θ is a highest root of \mathcal{G} , k - an integer called a level of an RSOS. A remarkable fact is that a critical (multicritical) behaviour of RSOS is described by some rational $2D$ CFT, namely, by a minimal models with the W - symmetry [4], or coset models

$$\mathcal{M}_{\mathcal{G},k} = \frac{\mathcal{G}_1 \times \mathcal{G}_k}{\mathcal{G}_{k+1}} \quad (2)$$

Another crucial observation [5] is that a scaling behaviour of an RSOS in a vicinity of a critical point is described by some (1+1) relativistic scattering theory which can be identified with a massive perturbation of the CFT by a relevant operator from the Hilbert space. These perturbations usually preserve an infinite number of charges. It means that the relativistic scattering theory is a factorized scattering theory (FST) [6] and it's S-matrix obeys the natural requirements: factorizability, crossing symmetry and unitarity. The RSOS S-matrices are the S-matrices of kinks, interpolating between different vacua of a massive theory [8, 9]. Remarkably, they are objects of the same nature as local probabilities of RSOS and can be thought of as some special limit of these probabilities (generally taken at another level) upon imposition of the above requirements. Exact solutions of RSOS are rarely achievable. Therefore, scaling solutions given by an S-matrix are very important and can be a subject of a further investigation by means of TBA [7]. The RSOS scattering theories have been intensively studied for the recent several years [8] - [14]. D. Gepner suggested a general classification program based on a fusion ring structure of underlying RCFT. However, the attention was paid mostly to the certain class of rational FST (RFST) corresponding to a "thermal" shift from the critical point

$$\mathcal{M}_{\mathcal{G},k}^{(t)} = \mathcal{M}_{\mathcal{G},k} + g \int \Phi_{\rho, \rho+\theta}. \quad (3)$$

Here

$$\rho = \sum \lambda_{fund} \quad (4)$$

and we are using notations for RCFT primaries introduced in [15]. Thermal deformation preserves a discrete group of original RSOS (the Weil group of \mathcal{G}). We briefly review these models in Sec.2

The exceptions are the $\Phi_{1,2}$ and $\Phi_{2,1}$ perturbations of minimal CFT destroying Z_2 symmetry considered by F. Smirnov [11] and the perturbations of critical parafermions by the first parafermionic current destroying Z_N (V. Fateev [16]). In both cases the interesting kink systems are constructed. In Sec.3 we present a large class of RFST including as the special cases the examples of F. Smirnov and V. Fateev. We show that there are two integrable primaries in each of the considered RCFT which violate the Weyl group of \mathcal{G} in a way similar to the Z_2 - violation by the magnetic field coupled to the spin density operator of the critical Ising model. In order to stress this analogy we call the two perturbations "electric" and "magnetic" and the resulting RCFT - $\mathcal{M}_{\mathcal{G},k}^{(e)}$ and $\mathcal{M}_{\mathcal{G},k}^{(h)}$ respectively

$$\mathcal{M}_{\mathcal{G},k}^{(e,h)} = \mathcal{M}_{\mathcal{G},k} + g \int \Phi^{(e,h)} \quad (5)$$

$$\Phi^{(h)} \equiv \Phi_{\rho, \rho+\omega} \quad (6)$$

$$\Phi^{(e)} \equiv \Phi_{\rho+\omega, \rho} \quad (7)$$

We denote by ω the highest weight of the vector representation of \mathcal{G} .

In Sec. 4 we investigate a quantum symmetry of the constructed RFST and argue that there is a twisted affine algebra attached to each of them according to some rule. It worthwhile mentioning that thermal models can be viewed as restricted complex ATFT obtained by the highest root affinization $\mathcal{G} \rightarrow \mathcal{G}^{(1)}$ as long as $\mathcal{M}^{(e,h)}$ correspond to non-simply-laced complex ATFT taken at rational points of a q-deformation parameter. A quantization of the complex ATFT for non-simply-laced algebras is a challenging problem [14]. For the real coupling constant the solution found in [17] and studied in [18] exhibits an interesting duality property. In a sense the S-matrix solution for $\mathcal{G} = SO(N)$ suggested in Sec. 5 is a solution for the complex non-simply-laced ATFT at "unitary" points and we hope that it will shed light on the general situation.

In Sec. 6 we compare the solution to the known answers at the points where our family of models intersects with some families of $\mathcal{M}^{(t)}$ - type.

Sec. 7 is devoted to discussions.

2 Thermal perturbations of minimal W-models and RSOS - scattering

Consider an RSOS on a 2D lattice whose critical regime is described by $\mathcal{M}_{\mathcal{G},k}$. For each link (ij) of the lattice the variables λ_i and λ_j obey the admissibility condition

expressed by the following relations

$$\begin{aligned}\pi_{\lambda_j} &\in \pi_{\lambda_i} \otimes \pi_1 \\ \pi_{\lambda_i} &\in \pi_{\lambda_j} \otimes \pi_{\bar{1}}\end{aligned}\tag{8}$$

ϕ_1 and $\pi_{\bar{1}}$ stand for fundamental and antifundamental representations of \mathcal{G} . If we denote by dots possible values of λ selected by (1) and connect by a link each pair of admissible λ 's then we obtain a graph G with a finite number of nodes. RSOS configurations are given by different embeddings of the lattice into this graph. Local probabilities $W\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \middle| u\right)$ attached to each plaquette of the lattice are functions of an embedding of this plaquette and a spectral parameter u . a, b, c, d mark vertices of G . An integrability condition of the RSOS is expressed by the star - triangle equation on the local probabilities

$$\begin{aligned}&\sum_b W\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \middle| u\right) W\left(\begin{smallmatrix} e & f \\ b & d \end{smallmatrix} \middle| v\right) W\left(\begin{smallmatrix} g & e \\ a & b \end{smallmatrix} \middle| u+v\right) \\ &= \sum_b W\left(\begin{smallmatrix} g & e \\ b & f \end{smallmatrix} \middle| u\right) W\left(\begin{smallmatrix} g & b \\ a & c \end{smallmatrix} \middle| v\right) W\left(\begin{smallmatrix} b & f \\ c & d \end{smallmatrix} \middle| u+v\right)\end{aligned}\tag{9}$$

Solutions of (9) are found for the A, B, C and D algebras [2]. In the last three cases it was implied that the graph G is generated by π_ω . A remarkable property of these solutions is their rotational symmetry

$$W\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix} \middle| u\right) = \left(\frac{\phi_a \phi_c}{\phi_b \phi_d}\right)^{-1/2} W\left(\begin{smallmatrix} d & a \\ c & b \end{smallmatrix} \middle| \Lambda - u\right),\tag{10}$$

where ϕ_λ stands for the quantum dimension of the representation π_λ at the level k . Λ is a crossing parameter depending on k and the Coxeter number $c_{\mathcal{G}}$. The rotational symmetry is closely related to the crossing symmetry in $2D$ scattering. Another property connected with FST is quasiunitarity

$$\sum_d W\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \middle| u\right) W\left(\begin{smallmatrix} d & b \\ c & e \end{smallmatrix} \middle| -u\right) = f(u)f(-u)\delta_{ae}\tag{11}$$

For the B, C and D algebras

$$f(u) = \frac{\sin(\omega - u)}{\sin \omega} \frac{\sin(\Lambda - u)}{\sin \Lambda}\tag{12}$$

$$\omega = \frac{i\pi}{c_{\mathcal{G}} + k}\tag{13}$$

$$\Lambda = \omega c_{\mathcal{G}}/2\tag{14}$$

The solutions of [2] form a family parametrized by the elliptic parameter P . In the limit $P \rightarrow \infty$ local probabilities become simple combinations of trigonometric functions and the system exhibits a critical behaviour. Small deviation from zero in the P - direction is described by the thermal perturbation (3) of a RCFT. The corresponding $2D$ field theory possess an infinite number of IM. Their spins measured by the operator $s = L_0 - \bar{L}_0$ from $2D$ conformal algebra follow Coxeter exponents of \mathcal{G} modulo $c_{\mathcal{G}}$ with some exceptions for low k . An S -matrix of kinks which interpolate between the vacua of the theory satisfies (9) with exchange $k \rightarrow k-1$. This S -matrix can be obtained from the local probabilities W by the well known procedure. It was done first in [8, 10, 9] for $\mathcal{G} = A_1$. The answer is the restricted Sine - Gordon S -matrix. V. Fateev and H. de Vega generalized this result for $\mathcal{G} = A_n$, $n \geq 2$. The work for the rest of the algebras has been done by D. Gepner [19, 20]. The C - algebras have been treated in [14]. An S -matrix element corresponding to a kink-kink scattering process

$$K_{ab} + K_{bd} \rightarrow K_{ac} + K_{cd} \quad (15)$$

is given by

$$S \left(\begin{array}{cc|c} a & b & \theta \\ c & d & \end{array} \right) = F(\theta\Lambda/i\pi) \left(\frac{\phi_a\phi_d}{\phi_b\phi_c} \right)^{\frac{\theta}{2\pi i}} W \left(\begin{array}{cc|c} a & b & \theta\Lambda/i\pi \\ c & d & \end{array} \right) \quad (16)$$

θ is a relative rapidity of the in-coming and out-coming kinks. And $F(u)$ is a minimal solution of the equations

$$F(u) = F(\Lambda - u) \quad (17)$$

$$\frac{F(u)F(-u)}{f(u)f(-u)} = 1 \quad (18)$$

The S -matrix constructed in such a way satisfies **crossing**

$$S \left(\begin{array}{cc|c} a & b & \theta \\ c & d & \end{array} \right) = S \left(\begin{array}{cc|c} c & a & i\pi - \theta \\ d & b & \end{array} \right) \quad (19)$$

and **unitarity**

$$\sum_d S \left(\begin{array}{cc|c} a & b & \theta \\ c & d & \end{array} \right) S \left(\begin{array}{cc|c} d & b & -\theta \\ c & e & \end{array} \right) = \delta_{ae} \quad (20)$$

The star-triangle equation is not violated by additional factors in (16).

3 Integrability of the vector perturbations of the $SO(N)$ RCFT

The thermal shifts along the $\Phi_{\rho, \rho+\theta}$ - direction do not exhaust all the integrable directions in a parametric space near the criticality. It turns out that for a sufficiently large class of RCFT one can point out other integrable deformations [21] corresponding to "external fields" (5). For $\mathcal{G} = SU(2), SU(3)$ and $SO(N)$ their anomalous dimensions are

SU(2):

$$\begin{aligned}\Delta_p^{1,2} &= \frac{1}{4}\left(1 - \frac{3}{p+1}\right) \\ \Delta_p^{2,1} &= \frac{1}{4}\left(1 + \frac{3}{p}\right)\end{aligned}\tag{21}$$

SU(3):

$$\begin{aligned}\Delta_p^{(h)} &= \frac{1}{3}\left(1 - \frac{4}{p+1}\right) \\ \Delta_p^{(e)} &= \frac{1}{3}\left(1 + \frac{4}{p}\right)\end{aligned}\tag{22}$$

SO(N):

$$\begin{aligned}\Delta_p^{(h)} &= \frac{1}{2}\left(1 - \frac{N-1}{p+1}\right) \\ \Delta_p^{(e)} &= \frac{1}{2}\left(1 + \frac{N-1}{p}\right)\end{aligned}\tag{23}$$

$$p \equiv c_{\mathcal{G}} + k$$

The integrability of the resulting massive models defined by (5) can be proven by the counting argument. To this purpose let us note that both operators - $\Phi^{(h)}$ and $\Phi^{(e)}$ - are the most relevant ones in the operator algebras generated by each of them. So, in order to establish integrability one needs just to compare multiplicities appearing in the expansions of the conformal characters $\mathcal{V}_0/\partial\mathcal{V}_0$ and $\mathcal{V}^{(e,h)}/\partial\mathcal{V}^{(e,h)}$

$$\chi_{\mathcal{V}_0/\partial\mathcal{V}_0}(q) = (1-q)\chi_{\mathcal{V}_0}(q) + q = \sum_s a_s q^s,$$

$$\chi_{\mathcal{V}^{(e,h)}/\partial\mathcal{V}^{(e,h)}}(q) = (1-q)\chi_{\mathcal{V}^{(e,h)}}(q) = \sum_s b_s^{(e,h)} q^s$$

For the characters of the operators (21 - 22) one has

$$a_6 = b_5 + 1\tag{24}$$

for (23)

$$a_4 = b_3^{(e,h)} + 1\tag{25}$$

This means the existence of the nontrivial IM of spin $s = 5$ and $s = 3$ respectively. This fact for $\Phi_{1,2}$ and $\Phi_{2,1}$ operators in unitary minimal models has been known long ago [5].

In a sense we consider the perturbations of RCFT generalizing the latter ones treated in [11]. The models $\mathcal{M}_{SO(4),k}^{(e,h)}$ are somewhat curious. In this case original CFT is nothing but a tensor product of two copies of a minimal model because $SO(4) = SU(2) \times SU(2)$. The perturbations which couple the two copies are

$$\Phi^{(h)} = \Phi_{1,2}^{(1)} \Phi_{1,2}^{(2)} \quad (26)$$

$$\Phi^{(e)} = \Phi_{2,1}^{(1)} \Phi_{2,1}^{(2)} \quad (27)$$

thus

$$\mathcal{M}_{SO(4),k}^{(e,h)} = \mathcal{M}_k^{(1)} + \mathcal{M}_k^{(2)} + g \int \Phi^{(e,h)} \quad (28)$$

Both of these models enjoy Z_2 symmetry permuting the first and the second copy. The conserved charge of spin $s = 3$ can be constructed explicitly starting from two holomorphic currents of spin $s = 4$ respecting this permutation

$$J_1(z) =: \left(T^{(1)}\right)^2 : + : \left(T^{(2)}\right)^2 : \quad (29)$$

$$J_2(z) =: T^{(1)} T^{(2)} : \quad (30)$$

These two currents are opposed by the only $s = 3$ non-derivative descendant of $\Phi^{(1)} \Phi^{(2)}$:

$$L_{-3} \Phi^{(1)} \Phi^{(2)} + \Phi^{(1)} L_{-3} \Phi^{(2)}.$$

This descendant drops out of the r.h.s. of conservation law of the following linear combination:

$$J(z) = J_1(z) + \frac{2(\Delta - 1)}{\Delta} J_2(z),$$

hence

$$\bar{\partial} J = g \partial \Lambda_2(z, \bar{z}), \quad (31)$$

where $\Delta = \Delta_{4,p}^{(e,h)}$ is given by (23).

Integrability of the $\mathcal{M}_{3,p}^{(h)}$ has been discovered in [23]. Examination of W - characters allows to conjecture that in general the models (21,22) exhibit IM's at spins

$$s = 1, 5, 7, 11, \dots$$

and the models (23) at spins

$$s = 1, 3, 5, 7, \dots$$

for sufficiently large p 's.

4 Quantum symmetry

In order to find an RSOS S-matrix for the integrable models constructed in Sec.3 one has to choose between the Yang - Baxter solutions to start from. These solutions are naturally marked by the affine Lie algebras [2]. So, we have to understand, which of the affine algebras correspond to our models. The question is resolved by the following

Theorem

*Consider a RCFT given by (2) and an integrable massive perturbation by a primary $\Phi_{\rho, \rho+\lambda}$ (or $\Phi_{\rho+\lambda, \rho}$). The resulting RFST corresponds to a q -deformed affine Lie algebra whose Dynkin diagram $D(\mathcal{G}, \lambda)$ can be obtained by attaching additional $-\lambda$ root to the finite diagram $D(\mathcal{G})$ and **inversion** of the arrow (if any) connecting this root with the finite part (fig.1)*

It should be noted that the theorem is closely related to the duality between a root system of a Toda lattice and a set of the nonlocal charges observed in [24]. We will skip a rigorous proof of this theorem presenting just several checks for it which are quite convincing however. The simplest check is to examine a symmetry of a model in the rational limit $p \rightarrow \infty$. In such a limit the nonlocal charges [25] become local IM and can be easily constructed.

$$\forall \mathcal{G}; \quad \lambda = \theta$$

This case corresponds to thermal RSOS.

$$D(\mathcal{G}, \theta) = D(\mathcal{G}^{(1)})$$

Therefore we have to deal with $\mathcal{G}^{(1)}$ S-matrices in agreement with [8, 12, 19, 20].

$$\mathcal{G} = \mathbf{A}_1; \quad \lambda = \omega$$

For this case we have

$$D(A_1, \omega) = D(A_2^{(2)}).$$

The inversion does not change the diagram (fig.1) So, the S-matrix solution of the model can be obtained from the $A_2^{(2)}$ solution of YB equation as it was done in [11, 26]. IM's generating $A_2^{(2)}$ in the rational limit are constructed in [21]. The nonlocal charges for $p < \infty$ [22] form the $A_{2,q}^{(2)}$ algebra at $q^p = 1$.

$$\mathcal{G} = \mathbf{A}_2; \quad \lambda = \omega$$

In this case (fig. 1)

$$D(A_2, \omega) = D(D_4^{(3)}).$$

In the rational limit the model coincides with the complex $A_2^{(1)}$ Toda model at the second reflectionless point $\Delta_{pert} = 1/3$ [21]. One can explicitly construct $D_4^{(3)}$ Noether charges acting on the 8-plet of the lightest particles: six solitons and antisolitons plus two breathers. (see [21] for details). The 8-plet transforms in the vector representation of the $D_4^{(3)}$.

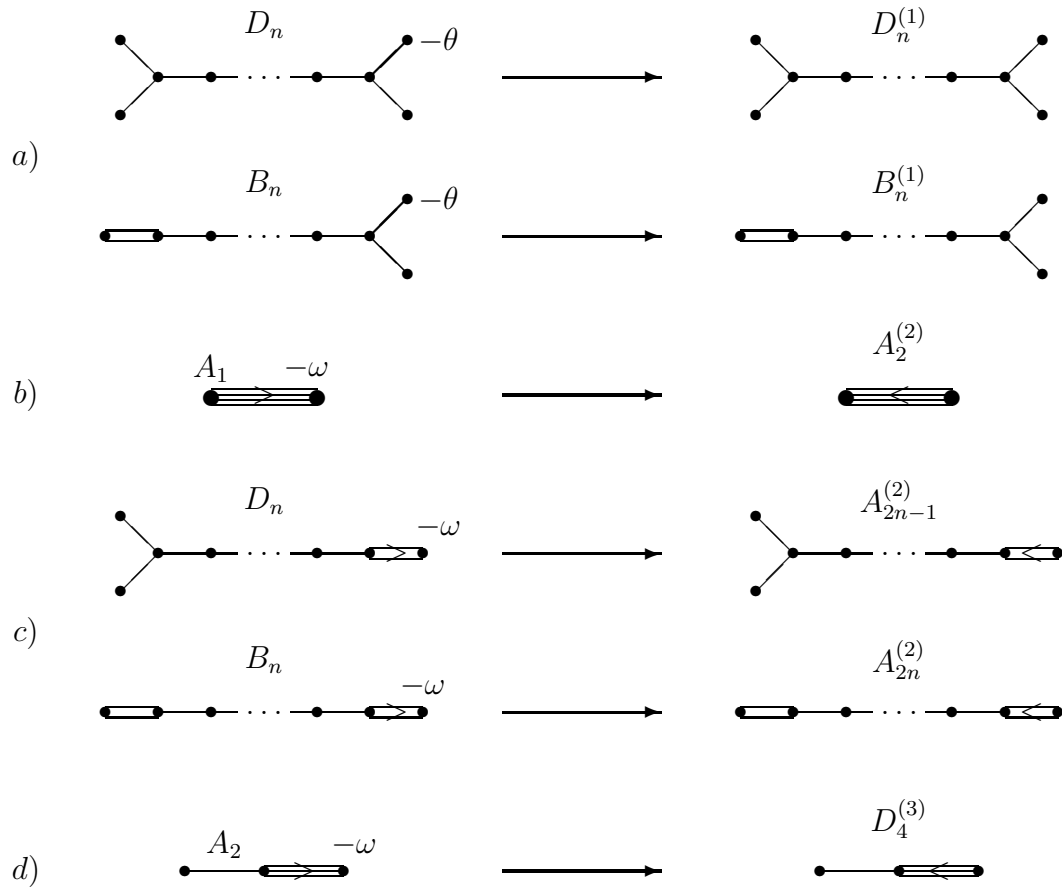


Figure 1:

- a) Thermal perturbation. No arrow inversion.
- b) $A_2^{(2)}$ - case. Inversion does not change the diagram.
- c) (e, h) - perturbations of $SO(N)$ theories.
- d) (e, h) - perturbations of $SU(3)$ theory.

$$\mathcal{G} = \mathbf{SO}(\mathbf{N}); \quad \lambda = \omega$$

This is the case to which the main attention is going to be paid in the rest of the paper.

$$D(SO(N), \omega) = D(A_{N-1}^{(2)}).$$

Once again it is very instructive to look at the rational limit of the model. We immediately see that in this limit the model is nothing else but N free massive fermions. As was observed in [27, 21] such a system exhibits the charges generating twisted affine algebra $A_{N-1}^{(2)}$.

So, the theorem agrees with the known cases. But a decisive support for it should be provided by the S-matrix construction of the next Section. This construction is done for the case $(SO(N), \omega)$. It is using the theorem as an input and shows up a perfect agreement with known results at all the checkpoints.

5 Fundamental kink-kink S-matrices

First, we construct an admissibility graph G describing the kink-kink scattering in the models (5) for $\mathcal{G} = SO(N)$. We will call them $\mathcal{M}_{N,p}^{(e,h)}$. The structure of this graph follows from the fusion algebra of original CFT. It is important to mention that

$$\Phi^{(e)}(z) \Phi^{(h)}(w) = \frac{\Phi_{\rho+\omega, \rho+\omega}}{z-w} \quad (32)$$

It means that in the $\mathcal{M}_{N,p}^{(e)}$ model an UV limit of the kink is given by the operator $\Phi^{(h)}$. The vacua of the theory correspond to the primaries from the fusion ring generated by $\Phi^{(h)}$. Two vacua are connected by a link if one of them can be obtained by the fusion of another one with the generating operator. The graph constructed in such a way describes also a finite ring structure of the representations of the group $SO(N)_q$ at

$$q = e^{-i\pi/p+1} \quad (33)$$

generated by tensoring of π_ω .

In the $\mathcal{M}_{N,p}^{(h)}$ model an UV limit of the kink is given by the operator $\Phi^{(e)}$ which defines a graph G . In this case it corresponds to a representation ring of $SO(N)_q$ at

$$q = e^{i\pi/p} \quad (34)$$

As an example in fig.2 we have drawn the admissibility graphs at levels $k = 1, 2$. It is important to note that fundamental kinks live in the *vector* representation of the quantum group in contrast with the thermal RSOS when kinks belong to *fundamental* (spinorial) representations. This is the main kinematic difference between $\mathcal{M}^{(e,h)}$ and $\mathcal{M}^{(t)}$. It can be viewed as two different ways to affinize $SO(N)$ algebras - the first one giving $A_{N-1}^{(2)}$ and the second $B^{(1)}, D^{(1)}$ respectively. So, the local probabilities of

[2] for $(SO(N), \omega)$ in the trigometric limit seem to be a suitable input for an $S^{(e,h)}$ -matrix construction. Of course, some changes are necessary. Otherwise, we will get just a subsector of a thermal theory with a spinor-antispinor sector missed [19]. It is natural to assume that the appropriate change of the crossing parameter Λ in the RSOS weights should be similar to the one in a vertex R-matrix case [28]. The right choice of the crossing parameter is given by

$$\Lambda_{N,p}^{(e,h)} = 1/2(N\omega^{(e,h)} + i\pi), \quad (35)$$

where

$$\omega^{(h)} = i\pi/p \quad (36)$$

$$\omega^{(e)} = -i\pi/p + 1 \quad (37)$$

The main ingredient of the S-matrix (16)- local probabilities W - should be borrowed from [2]:

$$W \left(\begin{array}{cc|c} a & a + \mu & u \\ a + \mu & a + 2\mu & \end{array} \right) = \frac{[\Lambda - u][\omega - u]}{[\Lambda][\omega]} (\mu \neq 0), \quad (38)$$

$$W \left(\begin{array}{cc|c} a & a + \mu & u \\ a + \mu & a + \nu + \mu & \end{array} \right) = \frac{[\Lambda - u][a_\mu - a_\nu + u]}{[\Lambda][a_\mu - a_\nu]} (\mu \neq \pm\nu), \quad (39)$$

$$W \left(\begin{array}{cc|c} a & a + \nu & u \\ a + \mu & a + \mu + \nu & \end{array} \right) = \frac{[\Lambda - u][u]}{[\Lambda][\omega]} \left(\frac{[a_\mu - a_\nu + \omega][a_\mu - a_\nu - \omega]}{[a_\mu - a_\nu]^2} \right)^{1/2} (\mu \neq \pm\nu), \quad (40)$$

$$W \left(\begin{array}{cc|c} a & a + \nu & u \\ a + \mu & a & \end{array} \right) = \frac{[a_\mu + a_\nu + \omega - \Lambda + u][u]}{[\Lambda][a_\mu + a_\nu + \omega]} \frac{\phi_a}{(\phi_{a+\mu}\phi_{a+\nu})^{1/2}} (\mu \neq \nu), \quad (41)$$

$$W \left(\begin{array}{cc|c} a & a + \mu & u \\ a + \mu & a & \end{array} \right) = \frac{[\Lambda + u][2a_\mu + \omega + 2\Lambda - u]}{[\Lambda][2a_\mu + \omega + 2\Lambda]} - \frac{[u][2a_\mu + \omega + \Lambda - u]}{[\Lambda][2a_\mu + \omega + 2\Lambda]} H_{a\mu}, \quad (42)$$

$$= \frac{[\Lambda + u][2a_\mu + \omega + u]}{[\Lambda][2a_\mu + \omega]} - \frac{[u][2a_\mu + \omega - \Lambda + u]}{[\Lambda][2a_\mu + \omega]} \frac{\phi_a}{\phi_{a+\mu}} (\mu \neq 0), \quad (43)$$

Here $[u] \equiv \sinh u$; μ, ν are weights from π_ω ,

$$a_\mu = [(a + \rho, \mu) - \frac{1}{2}\delta_{0,\mu}]\omega$$

and

$$H_{a\mu} = \sum_{\nu \neq \mu} \frac{[a_\nu + a_\mu + \omega + 2\Lambda]}{[a_\nu + a_\mu + \omega]} \frac{\phi_a}{\phi_{a+\nu}}.$$

In the last formula it is implied that a is admissible with $a + \nu$.

Now one has to solve the equation (18) in order to find a function $F(u)$. This function contains all the information about an analytic structure of an S-matrix. As usually, a solution of (18) is not unique. Namely, one can multiply it by any CDD - factor. In order to remove this ambiguity one should apply a minimality principle [29]. A combination of this principle with specialities of our problem gives the following answer

$$F(\theta) = \frac{\sinh \omega \sinh \Lambda}{\pi^2} Q(\theta) \prod_{n=1}^{\infty} \frac{Q(i\pi n + (-1)^n \theta)}{Q(i\pi n - (-1)^n \theta)}, \quad (44)$$

$$Q(\theta) \equiv \Gamma\left[\frac{\Lambda}{\pi^2}(\alpha - \theta)\right]\Gamma\left[1 - \frac{\Lambda}{\pi^2}(\alpha + \theta)\right]\Gamma\left[\frac{\Lambda}{\pi^2}(\beta - \theta)\right]\Gamma\left[1 - \frac{\Lambda}{\pi^2}(\beta + \theta)\right] \quad (45)$$

$$\alpha_{N,p}^{(e,h)} \equiv i\pi\omega_{N,p}^{(e,h)} / \Lambda_{N,p}^{(e,h)}, \quad (46)$$

$$\beta_{N,p}^{(e,h)} \equiv i\pi + \pi^2 / \Lambda_{N,p}^{(e,h)}. \quad (47)$$

The final answer is given by (16), (35) - (47). Let us note that with the only exception ($\mathcal{M}_{N,N-1}^{(e,h)}$) the S-matrix has two poles in the physical sheet at the points α and $i\pi - \alpha$. The first of them corresponds to a kink bound state in the s -channel whose mass divided by the mass of the fundamental kink

$$\gamma = 2 \cos \alpha_{N,p}^{(e,h)} / 2 \quad (48)$$

depends on p . This is the main difference of the theories in external fields from the thermal ones. In the thermal theories γ is stable [19]. This observation is connected with a nonrenormalization property of a mass ratio's in the $\mathcal{G}^{(1)}$ -type ATFT and the opposite property of the non-simply-laced or twisted ATFT.

6 Comparison with the known results and further examples

Among RCFT of $SO(N)$ -type there are three well-known models. The first one - $\mathcal{M}_{N,N-1}$ - lies on the critical line in the parametric space of Ashkin-Teller model and can be described by the free massless scalar field $\phi(z, \bar{z})$ compactified on orbifold. The second one - $\mathcal{M}_{N,N}$ - coincides modulo some irrelevant subtleties with Z_{2N} parafermionic theory. Finally, the third one - $\mathcal{M}_{N,\infty}$ - is the model of N free fermions. It gives us a nice opportunity to check our formulas comparing them to the results for integrable deformations of these models.

$$\mathcal{M}_{N,N-1}^{(e)}$$

The primary field $\Phi_{N,N-1}^{(e)}$ is marginal and coincides with U(1) current-current operator

$$\Phi_{N,N-1}^{(e)} = \partial\phi\bar{\partial}\phi.$$

The perturbation does not shift the theory from the critical point. It should mean that the S-matrix exists only in conformal limit $\theta \rightarrow \infty$. For the crossing parameter we have (35)

$$\Lambda_{N,N-1}^{(e)} = 0.$$

Therefore rapidity $\theta = i\pi u/\Lambda$ becomes infinite as it should in agreement with the above observation.

$$\mathcal{M}_{N,N-1}^{(h)}$$

This model corresponds to $q = e^{i\pi/N-1}$, hence only two representations - π_0 and π_ω - are allowed by the selection rule (1). The admissibility graph (fig.2a)) consists of two points connected by a link. It defines a Z_2 fusion ring

$$\begin{aligned}\pi_0 \otimes \pi_0 &= \pi_0 \\ \pi_0 \otimes \pi_\omega &= \pi_\omega \\ \pi_\omega \otimes \pi_\omega &= \pi_0 \\ \phi_0 &= \phi_\omega = 1\end{aligned}$$

It means that a fundamental kink is effectively equivalent to a scalar particle [8]. The UV-limit of this particle is given by $\Phi^{(e)} = \partial\phi\bar{\partial}\phi$. In other words the fundamental particle S-matrix coincides with that for the lightest breather from the SG model at

$$\beta^2/8\pi = \dim \Phi_{N,N-1}^{(h)} = 1/2N.$$

The general answer for this case reduces to

$$\begin{aligned}S(\theta) &= F_{N,N-1}^{(h)}(\theta) W \left(\begin{array}{cc|c} 0 & \omega & \Lambda\theta/i\pi \\ \omega & 0 & \end{array} \right) \\ W \left(\begin{array}{cc|c} 0 & \omega & \Lambda\theta/i\pi \\ \omega & 0 & \end{array} \right) &= -\frac{\sinh\left(\frac{\pi i}{N-1} - \frac{2N-1}{2N-2}\theta\right) \sinh\left(\frac{2N-3}{2N-2}\pi i - \frac{2N-1}{2N-2}\theta\right)}{\sin\frac{\pi}{N-1} \sin\frac{\pi}{2N-2}}\end{aligned}\tag{49}$$

Analytic structure of the F -function is somewhat special. Namely, besides the ordinary simple poles in the physical sheet at $\theta = \alpha, i\pi - \alpha$ there are two additional ones at $\theta = \beta, i\pi - \beta$, where

$$\begin{aligned}\alpha_{N,N-1}^{(h)} &= \frac{2\pi i}{2N-1} \\ \beta_{N,N-1}^{(h)} &= \frac{\pi i}{2N-1}\end{aligned}\tag{50}$$

At α - poles the S-matrix reduces to a projector on adjoint representation of $SO(N)_q$. The cancellation of these poles by zero's of the W - function agrees with the absence

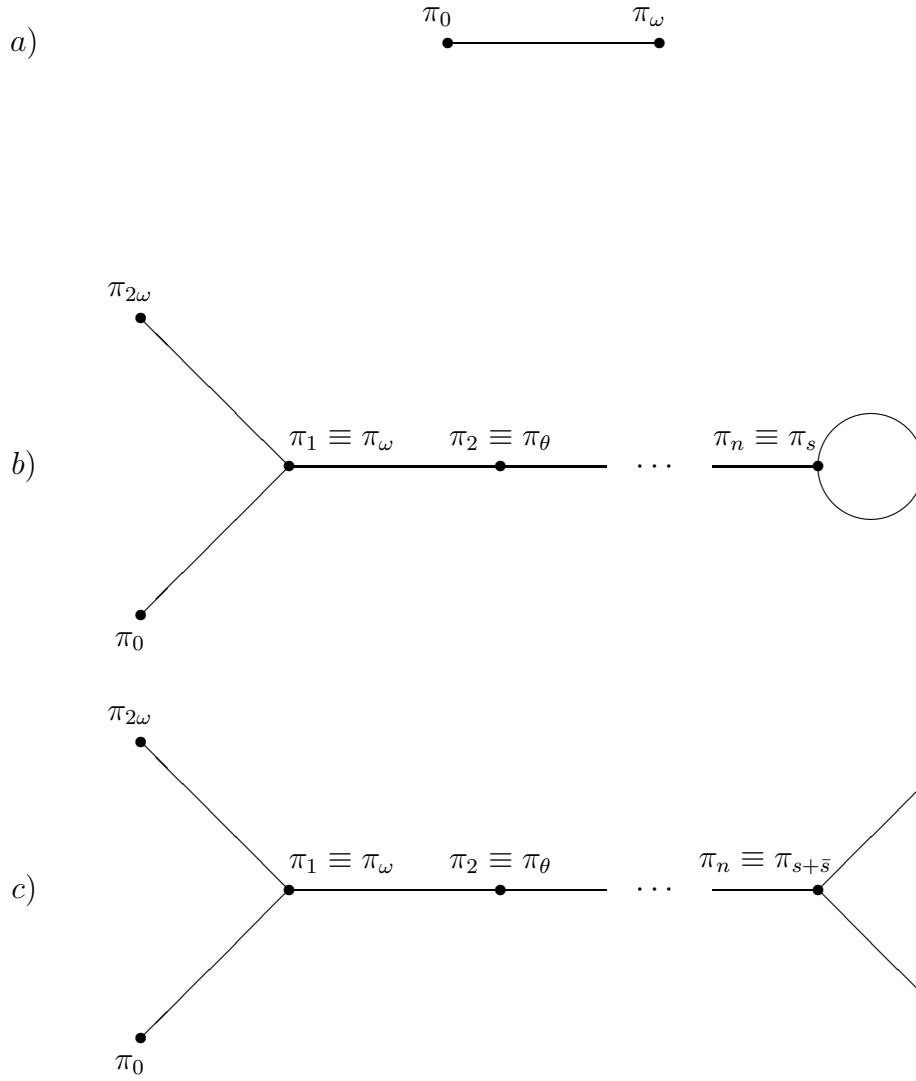


Figure 2: Admissibility graphs:

a) $G = SO(N)$, $k = 1$

b) $G = SO(2n + 1)$, $k = 2$

c) $G = SO(2n)$, $k = 2$

The b), c) graphs coincide with the McKay correspondence graphs of the dihedral groups d_{4n+2} , d_{4n} .

of π_θ in the fusion ring. So, the only poles are β -poles corresponding to a scalar bound state.

$$S_{N,N-1}^{(h)}(\theta) = \frac{\sinh \theta + i \sin \frac{\pi}{2N-1}}{\sinh \theta - i \sin \frac{\pi}{2N-1}}, \quad (51)$$

what coincides with the S-matrix for the lightest SG breather.

$$\mathcal{M}_{N,N}^{(e)}$$

This theory coincides with Z_{2N} parafermions perturbed by

$$\Phi_{N,N}^{(e)} = \psi_1 \bar{\psi}_1 + \psi_1^\dagger \bar{\psi}_1^\dagger,$$

where $\psi_1(z)$ denotes the first parafermionic current. Such a theory has been solved in [16]. This kink theory has no particle representation, therefore it can be defined by admissibility graph and corresponding W -functions in many equivalent ways. However, the analytic structure fixed by $F(\theta)$ does not depend on a representation. In our case the solution is given by (35 -47) with

$$\begin{aligned} \Lambda_{N,N}^{(e)} &= \frac{i\pi}{2N+2} \\ \omega_{N,N}^{(e)} &= -\frac{i\pi}{N+1} \end{aligned} \quad (52)$$

The F - function is equal to

$$F_{N,N}^{(e)}(\theta) = \frac{\sin \frac{\pi}{2N+2} \sin \frac{\pi}{N+1}}{\sinh \frac{i\pi-\theta}{2N+2} \sinh \frac{2\pi i+\theta}{2N+2}},$$

what exactly coincides with the unitarizing factor found in [16]. The admissibility graph for $N = 3$ is drawn in fig. 3.

$$\mathcal{M}_{N,N}^{(h)}$$

This theory coincides with thermalized parafermions described by the Koberle-Swieca S-matrix [30]. It is very interesting to see how this S-matrix follows from the general answer and we will discuss it in some more detail. The fusion ring is described by the admissibility graphs depicted in fig. 2b),c). In notations given on these pictures one has

$$\begin{aligned} N &= 2n + 1 \\ \pi_1 \otimes \pi_1 &= \pi_0 + \pi_{2\omega} + \pi_2 \\ \pi_2 \otimes \pi_1 &= \pi_1 + \pi_3 \\ \dots & \\ \pi_{n-1} \otimes \pi_1 &= \pi_{n-2} + \pi_n \\ \pi_n \otimes \pi_1 &= \pi_{n-1} + \pi_n \\ \pi_{2\omega} \otimes \pi_1 &= \pi_0 \otimes \pi_1 = \pi_1 \end{aligned} \quad (53)$$

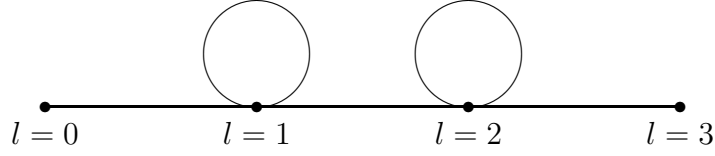


Figure 3: Admissibility graph for $G = SO(3)$ at level $k = 3$.

$$\begin{aligned}
N &= 2n \\
\pi_1 \otimes \pi_1 &= \pi_0 + \pi_{2\omega} + \pi_2 \\
\pi_2 \otimes \pi_1 &= \pi_1 + \pi_3 \\
&\dots \\
\pi_{n-2} \otimes \pi_1 &= \pi_{n-3} + \pi_{n-1} \\
\pi_{n-1} \otimes \pi_1 &= \pi_{n-2} + \pi_{2s} + \pi_{2\bar{s}} \\
\pi_{2\omega} \otimes \pi_1 &= \pi_0 \otimes \pi_1 = \pi_1 \\
\pi_{2s} \otimes \pi_1 &= \pi_{2\bar{s}} \otimes \pi_1 = \pi_{n-1},
\end{aligned} \tag{54}$$

where s and \bar{s} denote spinorial and antispinorial representations. Remarkable fact is that these fusion rings coincide with the ring of representations of the dihedral group d_{2N} . We will denote them by the same letters. The dihedral group is the symmetry group of an N -gon and consists of the $2N$ elements

$$d_{2N} \equiv [\epsilon_k, \bar{\epsilon}_k; k = 1, \dots, N].$$

The coincidence we are talking about is achieved provided

$$\begin{aligned}
\pi_1(\epsilon_1) &= \begin{pmatrix} e^{2\pi i/N} & 0 \\ 0 & e^{-2\pi i/N} \end{pmatrix} \\
\pi_1(\bar{\epsilon}_1) &= \begin{pmatrix} 0 & e^{2\pi i/N} \\ e^{-2\pi i/N} & 0 \end{pmatrix}
\end{aligned} \tag{55}$$

Dimensions of the representations of d_{2N} coincide with quantum dimensions of the corresponding representations of $SO(N)_q$ at level $k = 2$. Namely,

$$\begin{aligned}
\phi_1 &= \phi_2 = \dots = 2 \\
\phi_0 &= \phi_{2\omega} = \phi_{2s} = \phi_{2\bar{s}} = 1
\end{aligned} \tag{56}$$

It happens something very similar to the situation described in [11]. The $\lambda = 0$ component of the kink Hilbert space can be rearranged as a Hilbert space of particles and the fundamental kink behaves like a doublet of scalar particles forming the π_1 representation of d_{2N} . The poles of the S-matrix are placed at

$$\alpha_{N,N}^{(h)} = i\pi/N \quad (57)$$

and the S-matrix describing a scattering of the fundamental doublets is given by

$$\begin{aligned} S_{11}(\theta) &= S_{\bar{1}\bar{1}}(\theta) = \frac{\sinh(i\pi/2N + \theta/2)}{\sinh(i\pi/2N - \theta/2)} \\ S_{1\bar{1}}(\theta) &= S_{\bar{1}1}(\theta) = S_{11}(i\pi - \theta) \end{aligned} \quad (58)$$

in agreement with [30].

$$\mathcal{M}_{N,\infty}^{(e,h)}$$

In this limit both models - $\mathcal{M}^{(e)}$ and $\mathcal{M}^{(h)}$ - coincide with N free massive fermions, therefore the S-matrix should become trivial. Let us demonstrate that it also follows from the general answer. The $p \rightarrow \infty$ limit means that

$$\begin{aligned} \omega &\rightarrow 0 \\ \Lambda &\rightarrow i\pi/2 \\ a_\mu &\rightarrow 0 \\ a_\mu/\omega &\rightarrow \infty. \end{aligned} \quad (59)$$

The last relation is needed to neglect a "boundary effect" of the admissibility graph. Effectively it should look like an infinite N - dimensional cubic lattice. The quantum $SO(N)_q$ - group becomes classical and the kinks become vector $SO(N)$ - particles. Local probabilities W diverge in this limit and unitarizing function $F(\theta)$ vanishes in such a ratio that their product remains finite and diagonal. Namely,

$$S \left(\begin{array}{cc|c} a & b & \theta \\ c & d & \end{array} \right) = \delta_{a-c, b-d} \quad (60)$$

or in the vertex form

$$S_{ij}^{kl}(\theta) = \delta_{ik} \delta_{jl}. \quad (61)$$

This example concludes the checks of the S-matrix.

Let us pay some attention to $\mathcal{M}_{4,p}^{(e,h)}$ theories describing an integrable coupling of two minimal CFT (28). For example, $\mathcal{M}_{4,5}^{(e)}$ corresponds to bilayered Z_3 Potts models with the energy-energy coupling between the two layers. The kink - kink scattering for this physically interesting model immediately follows from the general answer (Sec. 5). The G - graph is presented on fig. 4, where we denote $SO(4)$ representation by quantum numbers of constituting $SU(2)$ representations. The S-matrix is given by (16),(35)-(47) with

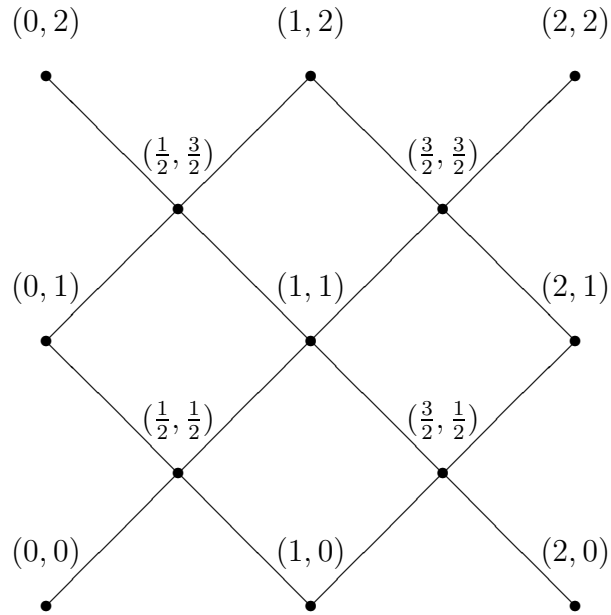


Figure 4: Admissibility graph for the kink theory describing two critical Z_3 - models coupled by the energy densities. The vacua are marked by highest weights of the $SO(4) = SU(2) \times SU(2)$ representations.

$$\begin{aligned}\omega &= -i\pi/6 \\ \Lambda &= i\pi/6.\end{aligned}\tag{62}$$

For this model $\phi_\omega = 3$ and quantum dimensions of all the allowed representations are integer. This is a hint that the model can be reconstructed as a particle scattering theory with fundamental particles forming a triplet. We hope to discuss this construction in a separate publication. It should be mentioned that in $\mathcal{M}^{(e)}$ - models α - pole leaves a physical sheet. Hence, no bound states appear in these models and the fundamental kink-kink S-matrix is complete.

7 Discussion

The main property of various integrable kink scattering theories is that in the rational limit they become particle theories described by some $2D$ integrable QFT's. These integrable QFT's are generally theories with an explicit symmetry under some finite group \mathcal{G} . Fundamental particles enter the theories by \mathcal{G} - multiplets and their S-matrix is of GN - type. For the $SU(N)$ - case such S-matrices have been suggested long ago [31]. In a sense the RSOS scattering theories could be viewed as appropriate restrictions of the GN - type models, or complex ATFT. The models suggested in the present paper are special, namely, they are restrictions of the trivially integrable QFT - free massive fermions. Nevertheless, restricted theories are highly nontrivial and exhibit a rich analytic structure. A natural question is whether these models correspond to a scaling limit of any statistical RSOS theories described by elliptic local probabilities. We think that those could be the recently constructed deluted RSOS models [32].

The S-matrix solution of the models $\mathcal{M}^{(e,h)}$ found in Sec. 5 presents an exact result for non-simply-laced complex ATFT at the discrete set of values of a coupling constant. A mass ratio of a fundamental kink and its bound state expressed by

$$\gamma = 2 \cosh \alpha(p, N)/2$$

depends on the coupling constant parameter p .

For example, if $N = 2n$ we should obtain a mass ratio of the first and the second soliton in $B_n^{(1)}$ ATFT described according to the theorem of Sec.4 by $A_{2n-1,q}^{(2)}$. This ratio predicted by our formula (48) is given by

$$\begin{aligned}m_2/m_1 &= \gamma = 2 \cos \frac{\pi}{2n+p}, \\ p &\equiv \frac{\beta^2}{4\pi^2 - \beta^2},\end{aligned}\tag{63}$$

where β is a coupling constant of an imaginary ATFT. This is in agreement with perturbative calculations [33, 34]. Of course, other solitonic mass ratios can be easily calculated along the same lines. By taking $N = 2n + 1$ one can obtain a mass spectrum of another set of non-simply-laced ATFT corresponding to affine superalgebras $A^{(4)}(0, 2n)$ described by the symmetry $B^{(1)}(0, n)_q$.

It should be noted that $\mathcal{M}^{(t)}$ - theories associated with $\mathcal{G}^{(1)}$ - like complex ATFT do not exhibit any coupling dependence of mass ratio's even if \mathcal{G} is non-simply-laced [19]. For such theories

$$\gamma = 2 \cosh \alpha(\mathcal{G}, p)/2 = 2 \cosh i\pi/c_{\mathcal{G}}$$

and does not depend on p . It happens due to a contribution of (para)fermionic loops which are always present in corresponding ATFT whenever \mathcal{G} is non-simply-laced [35]. In this sense the RSOS - solutions found here are truly non-simply-laced ATFT solutions with no fermions added to the action.

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